A NOTE ABOUT MINIMAL HYPERCONES

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ABSTRACT. This short note is concerned with a measure version criterion for hypersurfaces to be minimal. Certain natural flows and associated reflections for many minimal hypercones, including minimal isoparametric hypercones and area-minimizing hypercones, are studied.

1. INTRODUCTION

Various reflection principles play an important role in the research on minimal submanifolds, for instance, each connected part of the fixed point set of an isometry being totally geodesic, the construction of minimal hypersurfaces [8, 5] and etc. The paper [6] started the study of weakly reflective submanifolds (which have to be austere). Recently, [1] emphasized on the hypersurface case.

Definition 1.1 ([1]). *Let X^{n+1} be a complete Riemannian manifold and N^n an embedded hypersurface. Assume N divides X into two domains D_1 and D_2. Suppose that at any point p of N there is an isometry F of X such that*

$$F(p)=p,\quad F(N)=N,\quad F(D_1)=D_2,\quad F(D_2)=D_1.$$

Then N is called helicoidal in X.

It follows

Theorem 1.2 ([6, 1]). *Every helicoidal hypersurface is minimal.*

As an application, all Simons cones $C_{k,k}=C\left(S^k(\frac{1}{\sqrt{2}})\times S^k(\frac{1}{\sqrt{2}})\right)$ can be proved minimal. Note that $S^k(\frac{1}{\sqrt{2}})\times S^k(\frac{1}{\sqrt{2}})$ is helicoidal in $S^{2k+1}(1)$. However, Clifford tori of general type, namely,
$T_{k,l}=S^k\left(\sqrt{\frac{k}{k+l}}\right)\times S^l\left(\sqrt{\frac{l}{k+l}}\right)\subset S^{k+l+1}$ when $k\neq l$, or other minimal surfaces are not in general. Therefore, it is natural to detect situations for $N=C(T_{k,l})$ in $\mathbb{R}^{n+1}$.

We shall derive a measure version criterion for a hypersurface N to be minimal in §2. The formulations of our results take advantage of auxiliary foliation structures. Although interesting local versions can be gained, we wish to have “global" visions.

Given a flow with respect to suitable variable t along some foliation $\mathscr{F}=\{\mathscr{F}_p:\ p\in N\}$ perpendicular to $N=\{t=0\}$, if the (leafwise) associated reflection F along the foliation with respect to t by $(p,t)\mapsto(p,-t)$ in $\mathscr{F}_p$ satisfies:

(1). F preserves the volume form Ω_X of X and fixes N, and

(2). F sends the unit normal V_p of N at p to its antipodal,

then F is called a **perpendicular associated variable reflection** (PAVR) of $\mathscr{F}$ to N. However the PAVR structure alone is not sufficient for N to be minimal. It turns out that we need to consider the following quantity.

Definition 1.3. *Assume $W(p,t)$ is the velocity vector field of $\mathscr{F}_p$ in t. Then acceleration at p is defined to be*

$$I(p) = \frac{d\|W(p,t)\|}{dt}\Big|_{t=0}. \tag{1.1}$$

Remark 1.4. *One can define $I^{\pm}(\cdot)$ if W is C^1 merely in either side of N.*

Our criterion is

Theorem 1.5. *For a PAVR of N, $I(\cdot)$ vanishes identically in N if and only if N is minimal.*

Local PAVR structures always exist for minimal isoparametric hypercones. In §3 we shall establish "global" version.

Theorem 1.6. *For every isoparametric hypercone $N = C(M)$, there exist a PAVR defined almost everywhere on $\mathbb{R}^{n+1}$ with vanishing I in $N \sim 0$.*

The study of PAVR with unit length velocity vector field is closely related to stability and area-minimality. Among others, in particular, in §4 we recapture

Proposition 1.7. *Every minimal isoparametric hypercone with $n \geq 4g-1$ is stable in $\mathbb{R}^{n+1}$.*

If some preferred structure can exist in certain large domain, area-minimality follows (see [7]). Based on [2] the area-minimality property of a hypercone automatically induces

Proposition 1.8. *For each area-minimizing (regular) hypercone, there exist a natural PAVR structure defined almost everywhere with unit length velocity vector field.*

2. Proofs of Theorems 1.2, 1.5

Let us first give a proof for Theorem 1.2. Around any fixed $p \in N$, we choose an orientation, a unit normal vector field V of N and a local orthonormal frame $\{e_1, \cdots, e_n\}$ in N. We shall use the foliation $\mathscr{F}$ by exponential maps restricted to normals, i.e., geodesics determined by V, and variable t given by oriented distance to N. Then a local orthogonal frame $\{V, e_1, \cdots, e_n\}$ can be gained via parallel transport along these geodesics.

Proof of Theorem 1.2. Assume N divides X into two domains D_1 and D_2, and there is an isometry F of X such that $F(p) = p$, $F(N) = N$, $F(D_1) = D_2$, $F(D_2) = D_1$.

Let $\Omega = V^* \wedge e_1^* \wedge \cdots \wedge e_n^*$ and assume $F^*\Omega = \Omega$. Then

$$
\begin{aligned}
& F^*\left[d(i_V\Omega)\right] \\
&= F^*\left\{\Sigma_{i=1}^n e_i^* \wedge \Omega\left(\nabla_{e_i} V, \cdots\right) + V^* \wedge \Omega\left(\nabla_V V, \cdots\right)\right\} \\
&= F^*\left\{\left(\Sigma_{i=1}^n \langle \nabla_{e_i} V, e_i\rangle\right)\Omega\right\} \\
&= F^*\left\{-\Sigma_{i=1}^n \langle \nabla_{e_i} e_i,\ V\rangle\right\}\Omega \\
&= -F^*(H)\cdot\Omega
\end{aligned}
\tag{2.2}
$$

where $H(y)$ is the mean curvature function (with respect to V) of the integral hypersurface through y for the distribution $V^{\perp}$. Here we restrict ourself to the curve $\mathscr{F}_p$ and require $y \in \mathscr{F}_p$. It is obvious that $F(y) \in \mathscr{F}_p$ for y close to p.

On the other hand, since the vector field $(F_*)^{-1}V$ equals $-V$,

$$
\begin{aligned}
& F^*\left[d(i_V\Omega)\right] = d\left[F^*(i_V\Omega)\right] = d\left(i_{(F_*)^{-1}V} F^*\Omega\right) \\
&= \Sigma_{i=1}^n e_i^* \wedge \Omega\left(\nabla_{e_i}((F_*)^{-1}V), \cdots\right) + V^* \wedge \Omega\left(\nabla_V((F_*)^{-1}V), \cdots\right) \\
&= \left\{\Sigma_{i=1}^n \langle \nabla_{e_i}((F_*)^{-1}V), e_i\rangle + \langle \nabla_V((F_*)^{-1}V), V\rangle\right\}\Omega \\
&= H\cdot\Omega.
\end{aligned}
\tag{2.3}
$$

Thus (2.2) and (2.3) imply $H(p) = 0$. Similar argument works for the situation $F^*\Omega = -\Omega$ as well. ∎

Now we move to a similar proof for Theorem 1.5. A PAVR F along $\mathscr{F}$ with velocity vector field W with respect to variable t means that

$$F \circ F = \mathrm{id}, \tag{2.4}$$

$$F_* W(p,t) = -W(p,-t), \tag{2.5}$$

$$F^*\Omega = -\Omega. \tag{2.6}$$

Without loss of generality, we assume that W nowhere vanishes.

Proof of Theorem 1.5. Choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ in N and use parallel transport to extend it along $\mathscr{F}$. Set $V = \frac{W}{\|W\|}$. Then $\Omega = V^* \wedge e_1^* \wedge \cdots \wedge e_n^*$. Now (2.2) is replaced by

$$
\begin{aligned}
& F^*\left[d(i_W\Omega)\right] \\
&= F^*\left\{\Sigma_{i=1}^n e_i^* \wedge \Omega\left(\nabla_{e_i} W, \cdots\right) + V^* \wedge \Omega\left(\nabla_V W, \cdots\right)\right\} \\
&= F^*\left\{\left(\Sigma_{i=1}^n \langle \nabla_{e_i} W, e_i\rangle + \langle \nabla_V W, V\rangle\right)\Omega\right\} \\
&= F^*\left\{\Sigma_{i=1}^n \langle \nabla_{e_i} e_i,\ W\rangle - \langle \nabla_V W, V\rangle\right\}\Omega
\end{aligned}
\tag{2.7}
$$

and (2.3) by

$$
\begin{aligned}
&F^*\left[d(i_W\Omega)\right]=d\left[F^*(i_W\Omega)\right]=d\left(i_{F_*W}F^*\Omega\right)\\
=&\;-\left\{\Sigma_{i=1}^n e_i^*\wedge\Omega\left(\nabla_{e_i}(-W),\cdots\right)+V^*\wedge\Omega\left(\nabla_V(-W),\cdots\right)\right\}\\
=&\left\{\Sigma_{i=1}^n\langle\nabla_{e_i}W,e_i\rangle+\langle\nabla_V W,V\rangle\right\}\Omega\\
=&\left\{-\Sigma_{i=1}^n\langle\nabla_{e_i}e_i,\,W\rangle+\langle\nabla_V W,V\rangle\right\}\Omega
\end{aligned}
\tag{2.8}
$$

Hence (2.7) and (2.8) imply $\|W\|\cdot H-\nabla_V\|W\|=0$. Thus, in N,

$$
H\equiv 0\iff \mathrm{I}=\nabla_W\|W\|=\|W\|\cdot\nabla_V\|W\|=0. \tag{2.9}
$$

∎

Remark 2.1. *When $\|W\|\equiv 1$, W is at least C^0 in N and C^1 elsewhere, the proof also holds by limiting approach. So we can include such case in PAVR.*

3. About minimal isoparametric hypercones

3.1. **Basic Knowledge.** An isoparametric foliation on $S^n(1)$ is given by level sets of a smooth function $f:\ S^n\to\mathbb{R}$ with properties:

$$
\left\|\nabla f\right\|^2=b(f)
$$

and

$$
\Delta f=a(f)
$$

where $a,\ b$ are smooth functions from $f(S^n)$ to $\mathbb{R}$ and Δ is the Laplacian operator on S^n. For such structure the preimages $M_\pm$ of maximal and minimal values of f are minimal submanifolds of codimensions m_1+1 and m_2+1 respectively; and preimages of other values are all hypersurfaces with g distinct principal curvatures with alternating multiplicities $m_1,\ m_2,\ m_1,\cdots$. In particular each hypersurface leaf has constant mean curvature and $M_\pm$ are of distance $\frac{\pi}{g}$. Making use of the distance parameter θ from M_+ for hypersurface leaves, i.e., M_θ for the leaf of distance θ to M_+, we have the volume of M_θ

$$
A(\theta)=\mathrm{const}\cdot\left(\sin\frac{g\theta}{2}\right)^{m_1}\left(\cos\frac{g\theta}{2}\right)^{m_2} \tag{3.10}
$$

and therefore we derive

$$
H(\theta)=\begin{cases} m_1 g\cot(g\theta) & \text{for } g \text{ odd}\\ \dfrac{m_1 g}{2}\cot\dfrac{g\theta}{2}-\dfrac{m_2 g}{2}\tan\dfrac{g\theta}{2} & \text{for } g \text{ even}\end{cases} \tag{3.11}
$$

where our mean curvature H has length $\left\|\overrightarrow{H}\right\|$ and a sign with respect to $-\partial_\theta$ pointing to M_+.

3.2. **Construction of local PAVR.** Let $M = M_{\theta_0}$ be the unique isoparametric hypersurface of maximal volume. Now we shall search for a one-dimensional homothetic foliation $\mathscr{F}$ in some angular neighborhood of $C(M)$ in $\mathbb{R}^{n+1}$ of which each leaf curve perpendicularly intersects $C(M) \sim 0$ and the flow along $\mathscr{F}$ in the angle variable θ is $\Omega_{\mathbb{R}^{n+1}}$-measure invariant.

For $\overrightarrow{x} \in S^n$ (excluding focal submanifolds) with unique shortest arc $\widehat{xx_0}$ meeting M at $\overrightarrow{x}_0$ orthogonally, we shall rescale $\overrightarrow{x}$ to its appropriate multiple $r(\alpha)\cdot\overrightarrow{x}$ where α is the oriented arc length of $\widehat{xx_0}$. Then by homothety one constructs leaves in some angular neighborhood of ray $\{l\cdot\overrightarrow{x}_0 : l \in \mathbb{R}_+\}$ in span$\{\overrightarrow{x},\overrightarrow{x}_0\}$. Do the same procedure for all points of M.

Let us analyze quantities near $\overrightarrow{y} = r(\alpha)\cdot\overrightarrow{x}$ in the leaf $\mathscr{F}_{\overrightarrow{x}_0}$ through $\overrightarrow{x}_0$. Apparently, $(1+\Delta l)\cdot\overrightarrow{y}$ lies in the leaf $\mathscr{F}_{(1+\Delta l)\cdot\overrightarrow{x}_0}$. Hence the projection of $\Delta l\cdot\overrightarrow{y}$ to the n-plane $\left(T_{\overrightarrow{y}}\mathscr{F}_{\overrightarrow{x_0}}\right)^{\perp}$ orthogonal to $\mathscr{F}_{\overrightarrow{x_0}}$ has length

$$\left|\Delta l\right|\cdot r\cdot\frac{1}{\sqrt{1+(\frac{dr}{rd\alpha})^2}}.$$

Let F_t be the map moving points forward along $\mathscr{F}$ by angle t. Note that, corresponding to $\frac{d}{d\alpha}$,

$$W = dF_\alpha\left(\frac{d}{d\alpha}\Big|_{\overrightarrow{x}_0}\right) = \frac{d}{dt}\Big|_{t=0}F_t(y) \tag{3.12}$$

of length $\sqrt{r^2+(\frac{dr}{d\alpha})^2}$. Choose

$$r(\alpha) = \left(\frac{A(0)}{A(\alpha)}\right)^{\frac{1}{n+1}}, \tag{3.13}$$

where $A(\alpha)$ stands for the volume of $M_{\theta_0+\alpha}$. Then it follows, by the property of isoparametric foliation,

(3.14)

$$\left\|dF_\alpha\left(e_1\wedge\cdots\wedge e_{n-1}\wedge\left(\frac{d}{d\alpha}|_{\overrightarrow{x}_0}\right)\wedge\overrightarrow{x}_0\right)\right\| = r^{n-1}\frac{A(\alpha)}{A(0)}\cdot\sqrt{r^2+\left(\frac{dr}{d\alpha}\right)^2}\cdot\frac{r}{\sqrt{1+\left(\frac{dr}{rd\alpha}\right)^2}} = 1$$

where $\{e_1,\cdots e_{n-1}\}$ is an orthonormal basis of $T_{\overrightarrow{x}_0}(M)$. In such way we obtain an $\Omega_{\mathbb{R}^{n+1}}$-invariant flow F_t and the associated reflection along $\mathscr{F}$ with respect to angle is an PAVR by (3.13) and $\|W\| = \sqrt{r^2+(\frac{dr}{d\alpha})^2}$.

Remark 3.1. *Although the measure preserving reflection in angle exists around every isoparametric hypercone, only the one centered at $C(M)\sim 0$ is PAVR.*

It is not hard to see that if one uses parameter $w = w(\alpha)$, e.g. w can be mean curvature $H(\alpha)$ of the corresponding isoparametric hypersurface, for an Ω-invariant flow in variable w, then the corresponding model will be

$$r(\alpha) = \left(\frac{K(0)A(0)}{K(\alpha)A(\alpha)}\right)^{\frac{1}{n+1}} \tag{3.15}$$

where $K(\alpha)=\left(\frac{dw}{d\alpha}\right)^{-1}$. Since $\frac{d}{dw}=K(\alpha)\frac{d}{d\alpha}$, one can get some similar relation to (3.14).

Another way to see this is from the following equivalent construction for C^∞ divergence-free homothetic vector fields. For a homothetic vector field

$$W_{(R,\alpha)}=\frac{K(\alpha)\cdot R}{r(\alpha)}\cdot\left(r(\alpha)\partial_\alpha+\frac{dr(\alpha)}{d\alpha}\partial_R\right) \tag{3.16}$$

in the "polar" coordinate (R,α) where $\partial_\alpha=\frac{1}{R}\cdot\frac{d}{d\alpha}$ of unit length, we have

$$\begin{aligned}\frac{L_W(\Omega)}{\Omega}=&\ \mathrm{div}(W)\\
=&\ K(\alpha)\cdot\mathrm{div}\left[\frac{R}{r}\cdot\left(r\partial_\alpha+\frac{dr}{d\alpha}\partial_R\right)\right]+\partial_\alpha(K)\cdot R\\
=&\ K(\alpha)\cdot\left\{\frac{R}{r}\cdot\mathrm{div}\left[\left(r\partial_\alpha+\frac{dr}{d\alpha}\partial_R\right)\right]+\partial_\alpha\left(\frac{R}{r}\right)\cdot r+\partial_R\left(\frac{R}{r}\right)\cdot\frac{dr}{d\alpha}\right\}+\partial_\alpha(K)\cdot R\\
=&\ K(\alpha)\cdot\left\{\frac{R}{r}\cdot\left[r\cdot\mathrm{div}(\partial_\alpha)+\partial_\alpha(r)+\frac{dr}{d\alpha}\cdot\mathrm{div}(\partial_R)\right]-\frac{1}{r}\cdot\frac{dr}{d\alpha}+\frac{1}{r}\cdot\frac{dr}{d\alpha}\right\}+\frac{dK}{d\alpha}\\
=&\ K(\alpha)\cdot\left\{\frac{R}{r}\cdot\left[r\cdot\frac{H}{R}+\frac{dr}{d\alpha}\cdot\frac{1}{R}+\frac{dr}{d\alpha}\cdot\frac{n}{R}\right]\right\}+\frac{dK}{d\alpha}\\
=&\ K(\alpha)\cdot\left\{H(\alpha)+\frac{n+1}{r(\alpha)}\cdot\frac{dr(\alpha)}{d\alpha}\right\}+\frac{dK}{d\alpha}.\end{aligned} \tag{3.17}$$

For $L_W(\Omega)$ to be zero everywhere, we have

$$r(\alpha)=\left(\frac{K(0)}{K(\alpha)}\right)^{\frac{1}{n+1}}\cdot\exp\left(\frac{-1}{n+1}\int_0^\alpha H(s)ds\right) \tag{3.18}$$

with $r(0)=1$. With the aid of (3.10) and (3.11), solution (3.18) is exactly

$$r(\alpha)=\left(\frac{K(0)}{K(\alpha)}\cdot\frac{A(0)}{A(\alpha)}\right)^{\frac{1}{n+1}}. \tag{3.19}$$

Remark 3.2. *Only when $\frac{d(K\cdot A)}{d\alpha}\Big|_\alpha=0$, the associated reflection is a PAVR for the cone corresponding to α. If one takes $K(\alpha)=\frac{1}{A(\alpha)}$, then $r\equiv 1$ and consequently the flow restricted to S^n is Ω_{S^n}-measure invariant. In such way a local PAVR of M in S^n is obtained.*

It is easy to see that, for the case of odd g, the minimal isoparametric hypersurface has $\theta_0=\frac{\pi}{2g}$ and it sits precisely at the middle between focal submanifolds. Therefore, the PAVR with $K(\alpha)=\frac{1}{A(\alpha)}$ can be defined almost everywhere on $\mathbb{R}^{n+1}$, namely on $\mathbb{R}^{n+1}\sim\left(C(M_+)\bigcup C(M_-)\right)$, and exchanges two chambers divided by $C(M)$.

3.3. **For global PAVR.** For even g, M may have different distances to focal submanifolds in general and thus the preceding PAVR cannot extend to almost everywhere. We shall show how to construct global PAVRs with vanishing I in the minimal isoparametric cone.

Now shift to variable θ the distance to M_+ and apply further modifications. Since

$$\frac{d}{d\theta}H=-\frac{g^2}{4}\left(\frac{m_1}{s^2}+\frac{m_2}{c^2}\right)<0 \tag{3.20}$$

where $s=\sin\frac{g\theta}{2}$ and $c=\cos\frac{g\theta}{2}$, we can write $\theta=\theta(H)$ with $\theta_0=\theta(0)=\frac{2}{g}\arctan\sqrt{\frac{m_1}{m_2}}$. Thus

$$\dot{\theta}=\frac{d\theta}{dH}=-\frac{4}{g^2}\left(\frac{m_1}{s^2}+\frac{m_2}{c^2}\right)^{-1} \tag{3.21}$$

For each value H, set $\theta_1=\theta(H)$ and $\theta_2=\theta(-H)$. It follows from (3.11) that for every H

$$\tan\frac{g\theta_1}{2}\cdot\tan\frac{g\theta_2}{2}=\frac{m_1}{m_2}, \tag{3.22}$$

and

$$-\frac{\dot{\theta}_2}{s_2c_2}=\frac{\dot{\theta}_1}{s_1c_1}. \tag{3.23}$$

So if we define

$$T(H)\triangleq-\frac{\dot{\theta}}{\sin\big(\theta(H)\big)\cos\big(\theta(H)\big)}=\frac{4}{g^2sc}\left(\frac{m_1}{s^2}+\frac{m_2}{c^2}\right)^{-1}\rangle 0, \tag{3.24}$$

then (3.23) leads to

$$T(H)=T(-H). \tag{3.25}$$

Introduce

$$h(H)=\int_0^H-\sin\big(\theta(S)\big)\cos\big(\theta(S)\big)dS=\int_{\theta_0}^{\theta(H)}-\frac{sc}{\dot{\theta}}d\theta=\int_{\theta_0}^{\theta(H)}\frac{1}{T}d\theta. \tag{3.26}$$

Then $\lim_{H\to\pm\infty}h(H)=\mp\infty$. Since

$$\frac{d}{dh}=(\frac{dh}{d\theta})^{-1}\cdot\frac{d}{d\theta}=T\cdot\frac{d}{d\theta}, \tag{3.27}$$

we construct

$$r(\theta)=\left(\frac{T(\theta_0)}{T(\theta)}\cdot\frac{A(\theta_0)}{A(\theta)}\right)^{\frac{1}{n+1}}, \tag{3.28}$$

for a homothetic foliation in which the flow in h is measure invariant. Combined with the fact that T attains its maximal at $H=0$, it follows by Remark (3.2) that choice (3.28) generates a PAVR (with variable h) defined on $\mathbb{R}^{n+1}\sim\big(C(M_+)\bigcup C(M_-)\big)$ with

$$I=\nabla_W\|W\|=\nabla_{T\cdot\frac{d}{d\theta}}\left\|T\cdot\frac{d}{d\theta}\right\|=T\left(\nabla_{\frac{d}{d\theta}}T\right)\left\|\frac{d}{d\theta}\right\|+T^2\nabla_{\frac{d}{d\theta}}\left\|\frac{d}{d\theta}\right\|=0 \tag{3.29}$$

in $C(M)\sim 0$.

Apparently, such a structure nicely exists on $\mathbb{R}^{n+1}\sim\big(C(M_+)\bigcup C(M_-)\big)$. Therefore we finish the proof of Theorem 1.6.

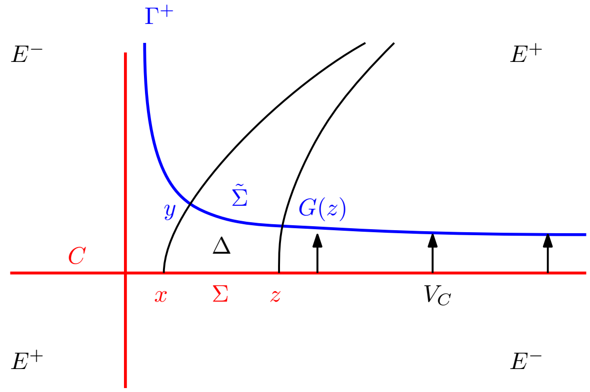

Illustration picture when $y \in \Gamma^+$

4. About area-minimizing hypercones

Given an area-minimizing (regular) hypercone $C(M)$. According to [2], there are unique smooth minimal hypersurface Γ^+ and Γ^- in chamber E^+ and E^- of $\mathbb{R}^{n+1} \sim C(M)$ respectively of unit distance to the origin so that their homotheties $\mathscr{E}$ foliate these two chambers. The associated perpendicular distribution forms a dilation-invariant foliation $\mathscr{F}$ of C^1-curves (smooth away from the cone). Assign the curves with the orientation pointing from E^- to E^+.

Proof of Proposition 1.8. By F_s we mean the moving forward along $\mathscr{F}$ by (signed) length s. Then the unit normal vector field to $\mathscr{E}$ is given by

$$V(\cdot) = \frac{d\big(F_s(\cdot)\big)}{ds}\Big|_{s=0} \tag{4.30}$$

Let $x \in C(M) \sim 0$, $y = F_s(x)$ and Γ_y be (smooth part of) the leaf through y of $\mathscr{E}$. For $z \in \Gamma_x$, define $s_{x,y}(z)$ to be the (signed) length of the curve segment in $\mathscr{F}_z$ connecting z and $\mathscr{F}_z \bigcap \Gamma_y$, such that

$$G(z) = F_{s_{x,y}(z)}(z) \in \Gamma_y. \tag{4.31}$$

Let $\gamma(t) \subset \big(\Gamma_x \bigcap \text{support of } \mathscr{F}\big)$ be a smooth curve with $\gamma(0) = x$ and $\gamma'(0) = Z$, and $s(t) = s_{x,y}\big(\gamma(t)\big)$. It is clear that

$$dF_s(Z) = \frac{d\left\{F_{s(t)}\big(\gamma(t)\big)\right\}}{dt}\Big|_{t=0} \tag{4.32}$$

and

$$dF_s(Z) - \langle dF_s(Z),\ V(y)\rangle \cdot V(y) = \frac{d\left\{F_{s(t)}\big(\gamma(t)\big)\right\}}{dt}\Big|_{t=0} = dG(Z). \tag{4.33}$$

Hence

$$dF_s\big(e_1 \wedge \cdots \wedge e_n \wedge V(x)\big) = dF_s\left(e_1 \wedge \cdots \wedge e_n\right) \wedge V(y) = dG\left(e_1 \wedge \cdots \wedge e_n\right) \wedge V(y). \tag{4.34}$$

where $\left\{e_1, \cdots, e_n\right\}$ form an orthonormal basis of $T_x\Gamma_x$.

For simplicity, assume x in some domain Σ in $C(M)\sim 0$, then applying Stokes' Theorem to $i_V\Omega$ over region Δ $\big($between Σ and $\tilde{\Sigma}=G(\Sigma)\big)$ produces that $\mathrm{vol}(\Sigma)=\mathrm{vol}(\tilde{\Sigma})$. As Σ shrinks to the point x, it leads to

$$\big\| dG(e_1\wedge\cdots\wedge e_n)\big\| = 1. \tag{4.35}$$

Therefore by (4.34) the length flow $\{F_s\}$ along $\mathscr{F}$ is Ω-invariant. This gives a PAVR with variable s (smooth away from the hypercone) with velocity vector field of unit length. ∎

Remark 4.1. *Let* $S^+\triangleq S^n(1)\bigcap\Gamma^+$ *and* $S^-\triangleq S^n(1)\bigcap\Gamma^-$*. Then the above PAVR is defined on* $\mathbb{R}^{n+1}\sim\big(C(S^+)\bigcup C(S^-)\big)$.

By the homothety of $\mathscr{F}$, V is translation-invariant in each ray through the origin in $\mathbb{R}^{n+1}$. For an area-minimizing isoparametric hypercone, V can be written as $V=\cos\big(\beta(\theta)\big)\,\partial_\theta+\sin\big(\beta(\theta)\big)\,\partial_R$. The $\Omega_{\mathbb{R}^{n+1}}$-invariance requirement

$$0=L_V\Omega=\frac{1}{R}(c\cdot H-s\cdot c\cdot\dot{\beta}+n\cdot s)\Omega \tag{4.36}$$

becomes

$$\dot{\beta}=\frac{H}{s}+\frac{n}{c}\ \text{ with }\ \beta(\theta_0)=0. \tag{4.37}$$

For $\dot{\beta}(\theta_0)$ to exist with value λ, it induces from (3.11) that

$$\lambda=n+\frac{-(n-1)\cdot g}{\lambda}, \tag{4.38}$$

with real solutions

$$\lambda_\pm=\frac{n\pm\sqrt{n^2-4g(n-1)}}{2} \tag{4.39}$$

only under the necessary condition

$$n^2-4g(n-1)\geq 0,\ \text{i.e.,}\ n+1\geq 4g. \tag{4.40}$$

It turns out that requirement condition (4.40) is in fact also sufficient (see page 44 of [7]). Based on the initial data at $\theta=\theta_0$, one can build up a divergent free vector field V of unit length in some angular neighborhood $\mathscr{N}$ (around $\theta=\theta_0$) of $C(M)$. It then follows that $i_V\Omega$ is a calibration form due to $\mathscr{E}$'s being foliation of minimal hypersurfaces (see [3, 4]). Since $V|_{C(M)\sim 0}$ is the unit normal vector field to $C(M)\sim 0$, $C(M)$ is area-minimizing in $\mathscr{N}$ by the fundamental theorem of calibrated geometry and therefore stable minimal. Hence we gain Proposition 1.7.

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